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On the Canonical Forms and Automorphs of Ternary Cubic Forms.

By L. E. Dickson.

Gordan has given* a complete set of canonical types of ternary cubic forms and has determined the algebraic irrationalities occurring in the reducing linear transformations. There does not seem to be at hand a reduction theory in which the coefficients of the form and those of the reducing transformations belong to a given field F. The case in which F has the modulus 3 is essentially different from the contrary case and will be treated in the present paper. After treating the reduction problem rationally in the initial field, we consider, in §§ 19–20, reductions involving irrationalities and obtain eleven ultimate canonical forms. This result for modular fields is in contrast to Gordan's results for the field of all complex numbers.

1. Let F be a field having modulus 3, and let

$$f \equiv \sum_{i}^{1,2,3} a_i x_i^3 + \sum_{i,j}^{1,2,3} c_{ij} x_i^2 x_j + b x_1 x_2 x_3 \tag{1}$$

have its coefficients in F. The Hessian of f is

$$\Sigma A_i x_i^3 - Q \{\Sigma c_{ij} x_i^2 x_j + b x_1 x_2 x_3\}, \tag{2}$$

where

$$Q \equiv b^2 - c_{21} c_{31} - c_{12} c_{32} - c_{13} c_{23}, \tag{3}$$

$$A_i \equiv c_{ij}^2 c_{ki} + c_{ik}^2 c_{ji} - b c_{ij} c_{ik} \qquad (i, j, k = 1, 2, 3). \tag{4}$$

We infer that Q is an invariant of f and that

$$\sum_{i}^{1,2,3} (A_i + Q a_i) x_i^3 \tag{5}$$

is a covariant of f. These facts indicate the exceptional character of the case in which the field has modulus 3.

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* *Transactions American Mathematical Society*, vol. 1 (1900), p. 403.

2. Suppose first that the c_{ij} are not all zero. By an evident transformation, we may set $c_{12}=1$. Applying in turn* B_{12t}, $B_{23t'}$, $B_{13t''}$, we may make $c_{21}=c_{13}=b=0$. The resulting form may be given the notation

$$f(x)\equiv x_1^2x_2-Qx_3^2x_2+Rx_2^2x_3+Sx_3^2x_1+\Sigma a_ix_i^3. \tag{6}$$

Under the transformation

$$x_1=y_1-tRy_3,\quad x_2=ty_1+y_2+t^2Ry_3,\quad x_3=y_3, \tag{7}$$

$f(x)$ becomes $f'(y)$, in which

$$\left\{\begin{array}{l} R'=R,\ S'=S-tQ+t^3R^2,\ a_1'=a_1+t^3a_2+t, \\ a_2'=a_2,\ a_3'=a_3+t^6R^3a_2-t^3R^3a_1-tRS-t^2QR-t^4R^3. \end{array}\right. \tag{8}$$

Henceforth, let F be the $GF[3^n]$. Then $S'=0$ requires

$$tQ-t^3R^2=S,\ t^3Q^3-t^9R^6=S^3,\ \ldots,\ t^{3^{n-1}}Q^{3^{n-1}}-tR^{2\cdot 3^{n-1}}=S^{3^{n-1}}. \tag{9}$$

The determinant of the coefficients of t, t^3, ..., $t^{3^{n-1}}$ equals

$$\Delta\equiv Q^{\frac{1}{2}(3^n-1)}-R^{3^n-1}. \tag{10}$$

If $\Delta\neq 0$, t, t^3, t^9, ... are uniquely determined by (9), and the resulting value of t^3 is seen to equal the cube of that of t, etc. Hence, if $\Delta\neq 0$, we may set $S=0$ in (6). Then, if $Q=0$, we multiply x_3 by a suitable mark and get

$$x_1^2x_2+x_2^2x_3+\Sigma a_ix_i^3. \tag{11}$$

But if $Q\neq 0$, we apply B_{32} and make $R=0$. Then if Q is the square of a mark τ, we introduce $x_1\pm\tau x_3$ and x_2 as new variables, and are led to the case treated in § 3. If Q is a not-square, we multiply x_1 by λ, and x_2 by λ^{-2}, and choose λ to specialize the new Q; there results

$$x_1^2x_2-\nu x_3^2x_2+\Sigma a_ix_i^3\quad (\nu \text{ a particular not-square}). \tag{12}$$

Next, let $\Delta=0$. If $R=S=0$, (6) becomes

$$x_1^2x_2+\Sigma a_ix_i^3. \tag{13}$$

If $R=0$, $S\neq 0$, we set $y_1=x_3$, $y_2=Sx_1$, $y_3=S^{-2}x_2$, and obtain (11). Finally, let $R\neq 0$, so that, by (10), Q is a square $\neq 0$. Multiplying x_1 by λ, x_2 by λ^{-2}, and x_3 by $\lambda\mu$, and taking† $\mu^2=Q^{-1}$, $\lambda^3=R\mu$, we get

$$x_1^2x_2-x_3^2x_2+x_2^2x_3+sx_3^2x_1+\Sigma a_ix_i^3. \tag{14}$$

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[*] In the usual notation, B_{12t} alters only x_1, replacing it by x_1+tx_2.

[†] Any mark ρ of the $GF[3^n]$ is the cube of the mark $\rho^{3^{n-1}}$.

In view of (8), this can be transformed into a similar form with

(e) $$s' = s - t + t^3.$$

Now $t^3 - t = c$ is solvable in the $GF[3^n]$ if and only if

$$\phi(c) \equiv c + c^3 + c^9 + \dots + c^{3^{n-1}}$$

vanishes. Hence, if $\phi(s) = 0$, (14) is equivalent to a similar form with $s = 0$. To the latter apply $B_{32,-1}$; there results $(y_1^2 - y_3^2)\,y_2 + \Sigma$, which obviously falls under the case treated in § 3. Since $\phi^3 = \phi$, there remains the case $\phi(s) = \pm 1$. Under the transformation $x_1' = -x_1$, the sign of s in (14) is changed. Hence we may set $\phi(s) = +1$. But if $\phi(s') = \phi(s)$, then $\phi(s' - s) = 0$, and (e) is solvable for t in the $GF[3^n]$. *Hence we may restrict s in* (14) *to be a particular solution of $\phi(s) = 1$.* In case n is prime to 3, we may set $s = 1$.

3. Next, let every c_{ij} be zero. According as $b \neq 0$ or $b = 0$, we get

(15) $$x_1 x_2 x_3 + \Sigma\, a_i x_i^3,$$

(16) $$\Sigma\, a_i x_i^3 \qquad (a_1,\ a_2,\ a_3 \text{ not all zero}).$$

4. *No form in one of the six systems* (11)–(16) *is reducible to a form in another of the systems by a ternary linear transformation in the $GF[3^n]$.*

Indeed, let f_1, f_2, f_3 denote the partial derivatives of f, given by (1). The number N of sets of solutions x_1, x_2, x_3 in the $GF[3^n]$ of $f_1 = f_2 = f_3 = 0$ is invariant under linear transformation.* For (11)–(16), we have $N = 3^n$, 3^n, 3^{2n}, 1, $3^{n+1} - 2$, 3^{3n}, respectively. The only case needing comment is (14). Eliminating x_1 between $f_1 = 0$ and $f_2 = 0$, we get $x_1^3 - x_1 x_3^2 - s x_3^3 = 0$. But $t^3 - t - s = 0$ is irreducible since $\phi(s) \neq 0$. Hence $x_1 = x_3 = 0$, and then $x_2 = 0$ by $f_3 = 0$.

For (11) and (12), we have the same value of N. But for (12), Q is a not-square ν; while for (11), (13) and (16), $Q = 0$; for (14) and (15), $Q = 1$. Under a transformation of determinant D, Q becomes $D^2 Q$, so that the quadratic character of Q is an invariant.

5. In view of § 4 the problem of the reduction to canonical forms falls into six independent problems. Consider the forms $S(a_1, a_2, a_3)$ of any one of the six systems, and let T be a transformation of one of its forms $S(a_1', a_2', a_3')$ into a second. Since the modulus is 3, T transforms $\Sigma\, b_i x_i^3$ into a similar sum $\Sigma\, \beta_i x_i^3$. *A system of forms is invariant under every transformation which replaces one of its forms by a second.*

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* There exists an invariant (other than Q) involving b and the c_{ij} alone.

We proceed to determine the groups of transformations leaving invariant the various systems of forms. As the usual direct method would be laborious, special devices have been invented.

6. *Group of the system* (11). Suppose that a particular form

$$F'_{a'_1 a'_2 a'_3} \equiv x_1'^2 x_2' + x_2'^2 x_3' + \Sigma a_i' x_i'^3$$

becomes $F_{a_1 a_2 a_3}$, given by (11), in view of the transformation

$$S:\ x_i' = \sum_{j=1}^{3} \alpha_{ij} x_j \qquad (i = 1, 2, 3), \tag{17}$$

with coefficients taken modulo 3. Then

$$\tfrac{1}{2}\frac{\partial^2 F}{\partial x_j \partial x_k} = x_1'(\alpha_{1j}\alpha_{2k} + \alpha_{1k}\alpha_{2j}) + x_2'(\alpha_{1j}\alpha_{1k} + \alpha_{2j}\alpha_{3k} + \alpha_{2k}\alpha_{3j}) + x_3'\alpha_{2j}\alpha_{2k} \equiv \psi_{jk},$$

since $\partial x_i' / \partial x_j = \alpha_{ij}$. Hence

$$0 = \psi_{13} = \psi_{33},\ x_1 = \psi_{12},\ x_2 = \psi_{11} = \psi_{23},\ x_3 = \psi_{22}.$$

From $0 \equiv \psi_{33}$, we get $\alpha_{23} = \alpha_{13} = 0$. Hence $\alpha_{33} \ddagger 0$, since $|\alpha| \ddagger 0$. Then $\alpha_{21} = 0$ from $0 \equiv \psi_{13}$. Next, $\psi_{11} = \alpha_{11}^2 x_2$, $\psi_{23} = \alpha_{22}\alpha_{33} x_2$. Hence $\alpha_{11}^2 = \alpha_{22}\alpha_{33}$, and

$$S = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ 0 & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{11}^2\alpha_{22}^{-1} \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} \alpha_{11}\alpha_{22} & \alpha_{11}\alpha_{12} + \alpha_{31}\alpha_{22} & 0 \\ 0 & \alpha_{11}^2 & 0 \\ -\alpha_{12}\alpha_{22} & \alpha_{12}^2 - \alpha_{22}\alpha_{32} & \alpha_{22}^2 \end{pmatrix}.$$

The necessary and sufficient conditions that $S^{-1}S = I$ are

$$\alpha_{11}^2\alpha_{22} = 1, \quad \alpha_{31} = \alpha_{11}^3\alpha_{12}.$$

Hence the transformations leaving system (11) invariant are

$$x_1' = \alpha x_1 + \alpha_{12}x_2,\ x_2' = \alpha^{-2}x_2,\ x_3' = \alpha^3\alpha_{12}x_1 + \alpha_{32}x_2 + \alpha^4 x_3 \qquad (\alpha \ddagger 0). \tag{18}$$

For the $GF[3^n]$, the order of the group is $3^{2n}(3^n - 1)$.

7. *Group of the system* (12). Evidently $(x_1^2 - \nu x_3^2)x_2$ is invariant under

$$T_{\alpha,\beta}:\ x_1' = \alpha x_1 + \beta x_3,\ x_3' = \nu^{-1}\beta x_1 + \alpha x_3,\ x_2' = x_2/(\alpha^2 - \nu^{-1}\beta^2), \tag{19}$$

where α and β are any marks not both zero, and under

$$L:\ x_3' = -x_3. \tag{20}$$

Let S, given by (17), be any transformation of the group of the system (12). We prove that S is generated by $T_{\alpha,\beta}$ and L. Now S and $ST_{0,1}$ do not both have $\alpha_{11} = \alpha_{13} = 0$; let S' denote the one with α_{11} and α_{13} not both zero. Then, for suitable marks α, β, $T_{\alpha,\beta}^{-1}S' \equiv S_1$ replaces x_1 by $x_1 - tx_2$. From the conditions that S_1 shall replace $x_1^2x_2 - \nu x_3^2 x_2$ by a form (12), we readily find that

S_1 is either the identity I or L. Now L transforms $T_{\alpha,\beta}$ into $T_{\alpha,-\beta}$. *Hence the group is composed of the* $2(3^{2n}-1)$ *transformations* $T_{\alpha,\beta}$ *and* $T_{\alpha,\beta}L$.

A simpler method of determining the group is to introduce the irrationality j, where $j^2=\nu$ defines the $GF[3^{2n}]$. Then $j^{3^n}=-j$. Introduce the conjugate variables

$$y=x_1+x_3j, \quad y^{3^n}=x_1-x_3j.$$

The form (12) becomes

$$yy^{3^n}x_2+a_2x_2^3+Ay^3+(Ay^3)^{3^n}, \quad A\equiv -a_1-a_3/j^3. \tag{21}$$

By inspection, or formally by § 10, this system of forms admits only the $2(3^{2n}-1)$ transformations T_t and T_tL, t any mark $\neq 0$ of the $GF[3^{2n}]$, where

$$T_t\colon\ y'=ty, \quad x_2'=t^{-3^n-1}x_2, \tag{19'}$$

$$L\colon\ y'=y^{3^n}. \tag{20'}$$

8. *Group of the system* (13). As in § 6, we get

$$\tfrac{1}{2}\frac{\partial^2 F}{\partial x_j\,\partial x_k}=x_1'(\alpha_{1j}\alpha_{2k}+\alpha_{1k}\alpha_{2j})+x_2'\alpha_{1j}\alpha_{1k}.$$

For $j=k=2$, we get $\alpha_{12}=0$. For $k=3$, we get $\alpha_{1j}\alpha_{23}+\alpha_{13}\alpha_{2j}=0$, $\alpha_{1j}\alpha_{13}=0$, for $j=1, 2, 3$. For $j=3$, the latter gives $\alpha_{13}=0$. Then $\alpha_{1j}\alpha_{23}=0$ $(j=1, 2, 3)$. Hence $\alpha_{23}=0$. For $j=1$, $k=1$ and 2, we get

$$x_2=-x_1'\alpha_{11}\alpha_{21}+x_2'\alpha_{11}^2, \quad x_1=x_1'\alpha_{11}\alpha_{22}.$$

Then $SS^{-1}=I$ requires that $\alpha_{11}^2\alpha_{22}=1$. *System* (13) *is invariant under exactly the* $3^{3n}(3^n-1)^2$ *transformations*

$$x_1'=\alpha_{11}x_1, \quad x_2'=\alpha_{21}x_1+\alpha_{11}^{-2}x_2, \quad x_3'=\alpha_{31}x_1+\alpha_{32}x_2+\alpha_{33}x_3 \quad (\alpha_{11}\neq 0,\ \alpha_{33}\neq 0). \tag{22}$$

9. *Group of the system* (14). This case is the most difficult of all and requires a new device. We introduce a cubic irrationality i such that the "determinant" of the enlarged field $F(i)$ yields a ternary cubic form belonging to the system (14); the factorization of this determinant in $F(i)$ is known.* For (14), $\phi(s)\neq 0$, so that x^3-x-s is irreducible in the $GF[3^n]$. Hence

$$i^3=i+s \tag{23}$$

defines the $GF[3^{3n}]$. To construct its determinant, expand†

$$(x_2+x_1i+x_3i^2)(y_2+y_1i+y_3i^2).$$

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* Dickson, *Transactions*, vol. 7 (1906), pp. 388, 389.

† The interchange of subscripts 1 and 2 is made here instead of later on.

The expansion may be exhibited with detached coefficients thus:

$$\begin{array}{c|lll} & y_1 & y_2 & y_3 \\ \hline 1 & s\,x_3 & x_2 & s\,x_1 \\ i & x_2 + x_3 & x_1 & x_1 + s\,x_3 \\ i^2 & x_1 & x_3 & x_2 + x_3\,. \end{array}$$

The determinant is congruent modulo 3 to

$$x_1^2 x_2 - x_3^2 x_2 + x_2^2 x_3 + s\,x_3^2 x_1 - s\,x_1^3 - x_2^3 - s^2 x_3^3\,. \tag{24}$$

This form belongs to the system (14); indeed, it is the only form (14) differing from its Hessian (2) only by a constant factor (here -1).

In view of its origin, (24) has the factors

$$\xi \equiv x_2 + x_1 i + x_3 i^2,\ \ \xi_1 \equiv x_2 + x_1 i^{3^n} + x_3 i^{2.3^n},\ \ \xi_2 \equiv x_2 + x_1 i^{3^{2n}} + x_3 i^{2.3^{2n}}. \tag{25}$$

It follows from the theory of conjugate variables that (14) equals

$$\xi\,\xi_1\,\xi_2 + \beta\,\xi^3 + \beta^{3^n}\xi_1^3 + \beta^{3^{2n}}\xi_2^3, \tag{26}$$

or, as we may write,

$$\xi^{1+3^n+3^{2n}} + \beta\,\xi^3 + (\beta\,\xi^3)^{3^n} + (\beta\,\xi^3)^{3^{2n}}. \tag{26$'$}$$

In case we wish to pass from the forms and transformations in the $GF[3^{3n}]$ to those in the $GF[3^n]$, we need the value of β, viz.,

$$\beta = -(a_1 + s)\,i^3 - (a_2 + 1)(i^6 - 1) - (a_3 + s^2). \tag{27}$$

On applying to (26′) the respective substitutions

$$\xi' = t\,\xi, \tag{28}$$

$$\xi' = t\,\xi^{3^n}, \qquad\qquad (t^{1+3^n+3^{2n}} = 1), \tag{29}$$

$$\xi' = t\,\xi^{3^{2n}}, \tag{30}$$

we obtain forms of type (26′) in which the new β's are

$$\beta\,t^3,\ \ \beta^{3^{2n}} t^{3^{2n+1}},\ \ \beta^{3^n} t^{3^{n+1}}. \tag{31}$$

Further, every automorph of the system (26), which preserves the conjugacy of the variables ξ, ξ_1, ξ_2, is given by (28), (29), or (30). *The group of the system* (14) *is of order* $3(1 + 3^n + 3^{2n})$; *it is generated by two operators* U *and* V *such that*

$$U^{1+3^n+3^{2n}} = I,\quad V^3 = I,\quad V^{-1}\,U\,V = U^{3^n}. \tag{32}$$

10. *Group of the system* (15). An immediate application of the method of § 6 shows that the $6(3^n - 1)^2$ transformations are

$$x_1' = \alpha\,x_i,\ \ x_2' = \beta\,x_j,\ \ x_3' = \alpha^{-1}\beta^{-1}x_k \qquad (i, j, k = 1, 2, 3). \tag{33}$$

11. The group of the system (16) is the general ternary group of order

$$\tau \equiv 3^{3n}(3^{3n}-1)(3^{2n}-1)(3^{n}-1). \tag{34}$$

12. As a check on the results in §§ 2–11, we note that

$$\frac{\tau\, 3^{3n}}{3^{2n}(3^n-1)}+\frac{\tau\, 3^{3n}}{2(3^{2n}-1)}+\frac{\tau\, 3^{3n}}{3^{3n}(3^n-1)^2}+\frac{\tau\, 3^{3n}}{3(1+3^n+3^{2n})}+\frac{\tau\, 3^{3n}}{6(3^n-1)^2}+\frac{\tau(3^{3n}-1)}{\tau}$$

equals $3^{10n}-1$, the total number of forms (1), not identically zero.

13. *Reduction of forms* (11). The available transformations are given by (18), which replaces (11) by a similar form with

$$a_1'=a_1\alpha^3+a_3\alpha^9\alpha_{12}^3,\quad a_2'=a_1\alpha_{12}^3+a_2\alpha^{-6}+a_3\alpha_{32}^3+\alpha^{-2}\alpha_{12}^2+\alpha^{-4}\alpha_{32},\quad a_3'=a_3\alpha^{12}. \tag{35}$$

Let first $a_3=0$. Since every mark is a cube, we may take $a_1=0$ or 1. Then for $\alpha=1$, $\alpha_{12}=0$, $\alpha_{32}=-a_2$, we have $a_1'=0$ or 1, $a_2=0$. We reach types 1 and 2 of the Table.

Let next $a_3\neq 0$. We may take $a_1=0$. Then for $\alpha_{12}=0$, $\alpha_{32}=\alpha^{-2}z$,

$$a_1'=0,\quad a_3'=\alpha^{12}a_3,\quad a_2'\alpha^6=a_2+a_3z^3+z. \tag{35'}$$

By choice of α and z, we can make $a_2'=0$ or 1, if $n>1$. Indeed, suppose that $\lambda\equiv a_2+a_3z^3+z$ is a not-square for every z in the $GF[3^n]$. Then

$$\lambda^{\frac{1}{2}(3^n-1)}\equiv\lambda\,\lambda^3\lambda^9\ldots\ldots\lambda^{3^{n-1}}=-1 \qquad \text{(for every } z\text{)}.$$

In the final factor, we replace z^{3^n} by z. Then the product is of degree

$$(3+9+27+\ldots\ldots+3^{n-1})+3^{n-1}\equiv\tfrac{1}{2}(5\,.\,3^{n-1}-3)<3^n.$$

Hence the relation must be an identity in z. The coefficient of the highest power of z is a_3^q, $q=1+3+\ldots\ldots3^{n-2}$, if $n>1$; but is a_3+1, if $n=1$. Excluding for the present the case $n=1$, we thus have $a_1=0$, $a_2=0$ or 1.

For $a_1=a_2=0$, $a_3\neq 0$, the only further normalization is the specialization of a_3' by the choice of α^{12}. Let ρ be a primitive root of the $GF[3^n]$. If n is odd, the even powers of ρ are 12th powers, since

$$\rho^2=\rho^2\rho^{3^n-1}=\{\rho^{\frac{1}{4}(3^n+1)}\}^4,\quad \tau=\{\tau^{3^{n-1}}\}^3;$$

while the odd powers of ρ are not-squares and hence not 12th powers. Hence for n odd, we may set $a_3'=\pm 1$, and reach types 3 and 4 of the Table. If n is even, the greatest common divisor of 12 and 3^n-1 is 4, so* that the only 12th powers are ρ^4, ρ^8, $\ldots$, $\rho^{3^n-1}\equiv 1$. Hence we may set $a_3'=1$, ρ, ρ^2, or ρ^3, and reach types 5 and 6 of the Table.

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* *Linear Groups* (Teubner, 1901), p. 45, § 63.

Next, let $a_1 = 0$, $a_2 = 1$, $a_3 = -a \neq 0$. We first seek the conditions under which this case can be reduced to the preceding. Such a reduction occurs if and only if

$$0 = 1 - a z^3 + z \equiv Z \tag{36}$$

is solvable for z in the $GF[3^n]$. Now

$$Z + a Z^3 + a^4 Z^9 + \dots + a^{1+3+3^2+\dots+3^{n-2}} Z^{3^{n-1}} = \psi(a) + z - a^{\frac{1}{2}(3^n-1)} z^{3^n} \tag{37}$$

where

$$\psi(a) \equiv 1 + a + a^4 + a^{13} + \dots + a^{1+3+3^2+\dots+3^{n-2}}. \tag{38}$$

Hence the equation $0 = Z$ is solvable for z in the $GF[3^n]$ only in the following cases: either a is a not-square, or a is a square such that $\psi(a) = 0$.

Let therefore a be a square such that $\psi(a) \neq 0$. We discuss the values taken by $a' = Z^2 a$, when z is chosen in the $GF[3^n]$ so that $Z \equiv 1 - a z^3 + z$ is a 6th power ($Z = \alpha^6$). Set $\psi = \psi(a)$, $\psi' = \psi(a')$. In view of (38), with a replaced by a', we get

$$Z\psi' = Z + a Z^3 + a^4 Z^9 + \dots + a^{1+3+\dots+3^{n-2}} Z^{3^{n-1}}. \tag{39}$$

Since a is a square and $z^{3^n} = z$, we deduce $Z\psi' = \psi$ from (37) and (39). To make Z a 6th power, it suffices to make it a square. Hence a necessary condition for the equivalence of two forms with the parameters a and a' is that ψ/ψ' be a square. We next show that this condition is sufficient. Let a and a' be given squares such that ψ/ψ' is a square. Now

$$\psi - a\psi^3 = 1 - a^{\frac{1}{2}(3^n-1)} = 0.$$

Hence $1 = a\psi^2$, $1 = a'\psi'^2$. Hence the equation $a' = Z^2 a$ is satisfied by the *square* $Z = \psi/\psi'$. For this value of Z, (37) and (39) give

$$0 = z - z^{3^n},$$

so that $Z = 1 - a z^3 + z$ is solvable for z in the $GF[3^n]$. *For the forms* (11) *in which* $a_1 = 0$, $a_2 = 1$, $a_3 = -a$, *a being a square* $\neq 0$ *such that* $\psi(a) \neq 0$, *those with* $\psi(a)$ *a square are equivalent, likewise those with* $\psi(a)$ *a not-square, while the two sets are not equivalent.* The two resulting canonical forms are listed under 7 in the Table.

It remains to treat the case $n = 1$, $a_3 \neq 0$, $a_1 = 0$. By (35), for $\alpha_{12} = 0$, $a_1' = 0$, $a_2' = a_2 + (a_3 + 1)\alpha_{32}$, $a_3' = a_3$. If $a_3 = 1$, we can make $a_2' = 0$, and reach type 3 of the Table. If $a_3 = -1$, no normalization is possible, and we have types 4 and 8 of the Table.

The determination of the automorphs of types 1–8 follows readily from (35).

Comment is needed only in the case of 7; viz., $a_1 = 0$, $a_2 = 1$, $a_3 = -a \neq 0$. Then

$$\alpha_{12} = 0, \ \alpha^4 = 1, \ 1 = \alpha^2 - a\,\alpha_{32}^3 + \alpha_{32}.$$

The case $\alpha^2 = 1$ leads to the automorphs listed. The case $\alpha^2 = -1$ is excluded, since $z \equiv \alpha_{32}$ would then be a root of (36), whereas the latter was shown to be not solvable when a is subject to the conditions on type 7.

14. *Reduction of forms* (12). The available transformations are $T_{\alpha,\beta}$ and $T_{\alpha,\beta}\,L$, given by (19) and (20). Now $T_{\alpha,\beta}$ replaces (12) by a similar form with

$$a_1' = a_1\alpha^3 + a_3\beta^3\nu^{-3}, \ a_3' = a_1\beta^3 + a_3\alpha^3, \ a_2' = a_2/(\alpha^2 - \nu^{-1}\beta^2)^3.$$

If a_1 and a_3 are not both zero, we can determine α and β to make $a_1' = 1$, $a_3' = 0$; the resulting type is 11 of the Table. If $a_1 = a_3 = 0$, we can make $a_2' = 0$ or 1; the resulting types are 9 and 10 of the Table. Since L leaves unaltered types 9, 10, 11, no two of them are equivalent.

15. *Reduction of forms* (13). Applying (22), we get

$$a_1' = \alpha_{11}^2\alpha_{21} + a_1\alpha_{11}^3 + a_2\alpha_{21}^3 + a_3\alpha_{31}^3, \ a_2' = a_2\alpha_{11}^{-6} + a_3\alpha_{32}^3, \ a_3' = a_3\alpha_{33}^3.$$

For $a_3 \neq 0$ we reach type 14 of the Table. For $a_3 = 0$, we first make $a_2 = 0$, -1, or $-\nu$, where ν is a particular not-square. For $a_3 = a_2 = 0$, we take $\alpha_{11} = 1$, $\alpha_{21} = -a_1$, and reach type 12. For $a_3 = 0$, $a_2 = -1$, we take $\alpha_{11} = \pm 1$ to preserve $a_2' = -1$. The problem of the specialization of $a_1' \equiv \pm a_1 + \alpha_{21} - \alpha_{21}^3$ is essentially the same problem as the specialization of s in (14). As at the end of § 2, we may take a_1' to be zero or a particular solution of $\phi(c) = 1$. The resulting types are 15 and 16 of the Tables.

Finally, for $a_3 = 0$, $a_2 = -\nu$, we take $\alpha_{11} = \pm 1$ to preserve $a_2' = -\nu$ Setting $\alpha_{21} = \pm z$, we have $\pm a_1' = a_1 + z - \nu z^3 \equiv W$. Then

$$W + \nu W^3 + \nu^4 W^9 + \ldots. + \nu^{1+3+\ldots.+3^{n-2}} W^{3^{n-1}} = \alpha + z - \nu^{\frac{1}{2}(3^n-1)} z^{3^n},$$

$$\alpha \equiv a_1 + \nu a_1^3 + \nu^4 a_1^9 + \ldots. + \nu^{1+3+\ldots.+3^{n-2}} a_1^{3^{n-1}}.$$

Since $\nu^{\frac{1}{2}(3^n-1)} = -1$, the equation $W = 0$ has the root* $z = \alpha$. Hence we may set $a_1' = 0$ and have type 13 of the Table.

16. The reduction of the forms (14)–(16) to the types 17–23 of the Tables offers no difficulty in view of the results of §§ 9–11.

17. In addition to the check in § 12 on the six systems, we note that the canonical forms within each system and their automorphs were checked by making two counts of the forms of the system.

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*The existence of a root also follows from a theorem on the analytic representation of substitutions, *Linear Groups*, § 81, p. 57.

	Canonical form	Automorphs	Number of automorphs
1	$A \equiv x_1^2 x_2 + x_2^2 x_3$	$(\alpha x_1 + \beta x_2,\ \alpha^{-2} x_2,\ \alpha^3 \beta x_1 - \alpha^2 \beta^2 x_2 + \alpha^4 x_3)$	$3^n (3^n - 1)$
2	$A + x_1^3$	$(x_1 + \beta x_2,\ x_2,\ \beta x_1 - (\beta^3 + \beta^2) x_2 + x_3)$	3^n
3	$A + x_3^3$ (n odd)	$(\pm x_1,\ x_2,\ x_3)$	2
4	$A - x_3^3$ (n odd)	$(\pm x_1,\ x_2,\ \beta x_2 + x_3),\ \beta^3 = \beta$	6
5	$A + \rho^i x_3^3$ (n even, $i = 0, 2$)	$(\alpha x_1,\ \alpha^2 x_2,\ \beta x_2 + x_3),\ \alpha^4 = 1,\ \rho^i \beta^3 = -\beta$	12
6	$A + \rho^i x_3^3$ (n even, $i = 1, 3$)	$(\alpha x_1,\ \alpha^2 x_2,\ x_3),\ \alpha^4 = 1$	4
7	$A + x_2^3 - a x_3^3 (n > 1)$	$(\pm x_1,\ x_2,\ \beta x_2 + x_3),\ a \beta^3 = \beta$	6
8	$A \pm x_2^3 - x_3^3 (n = 1)$	$(\pm x_1,\ x_2,\ \beta x_2 + x_3)$	6
9	$B \equiv x_1^2 x_2 - \nu x_3^2 x_2$	$T_{\alpha,\beta},\ T_{\alpha,\beta} L$ (§ 7)	$2 (3^{2n} - 1)$
10	$B + x_2^3$	$T_{\alpha,\beta},\ T_{\alpha,\beta} L,\ \alpha^2 - \nu^{-1} \beta^2 = 1$	$2 (3^n + 1)$
11	$B + x_1^3 + a_2 x_2^3$	$(x_1,\ x_2,\ \pm x_3)$	2
12	$x_1^2 x_2$	(22) with $\alpha_{21} = 0$	$3^{2n} (3^n - 1)^2$
13	$x_1^2 x_2 - \nu x_2^3$	$(\pm x_1,\ x_2,\ \beta x_1 + \gamma x_2 + \alpha x_3)$	$2 . 3^{2n} (3^n - 1)$
14	$x_1^2 x_2 + x_3^3$	$(\alpha x_1,\ \alpha^{-2} x_2 - \alpha^{-2} \beta^3 x_1,\ x_3 + \beta x_1)$	$3^n (3^n - 1)$
15	$x_1^2 x_2 - x_2^3$	$(\pm x_1,\ \beta x_1 + x_2,\ \gamma x_1 + \delta x_2 + \varepsilon x_3),\ \beta^3 = \beta$	$6 . 3^{2n} (3^n - 1)$
16	$x_1^2 x_2 - x_2^3 + a_1 x_1^3$	$(x_1,\ \beta x_1 + x_2,\ \gamma x_1 + \delta x_2 + \varepsilon x_3),\ \beta^3 = \beta$	$3 . 3^{2n} (3^n - 1)$
17	$(24) \sim (26)_{\beta \equiv 0}$	(28)-(30)	$3 (1 + 3^n + 3^{2n})$
18	$(14) \sim (26)_{\beta \neq 0}$	I, (29) with $t = \beta^{3^{n-1} - 3^{3n-1}}$ (30) with $t = \beta^{3^{2n-1} - 3^{3n-1}}$	3
19	$x_1 x_2 x_3$	(33)	$6 (3^n - 1)^2$
20	$x_1 x_2 x_3 + x_1^3$	$(x_1,\ \alpha x_i,\ \alpha^{-1} x_j),\ i, j = 2, 3$	$2 (3^n - 1)$
21	$x_1 x_2 x_3 + x_1^3 + x_2^3$	$I,\ (x_2,\ x_1,\ x_3)$	2
22	$x_1 x_2 x_3 + \alpha \Sigma x_i^3$	Permutations of $x_1,\ x_2,\ x_3$	6
23	x_1^3	$(x_1,\ \Sigma \alpha_j x_j,\ \Sigma \beta_j x_j)$	$3^{3n} (3^{2n} - 1) (3^n - 1)$

Specification of the parameters in the canonical forms.

5, 6	ρ is a fixed primitive root of the $GF[3^n]$.
7	a has two values, each a square in the $GF[3^n]$. For one, $\psi \equiv 1 + a + a^4 + a^{13} + \dots + a^{1+3+3^2+\dots+3^{n-2}}$ is a square; for the other, ψ is a not-square.
9, 13	ν is a fixed not-square in the $GF[3^n]$.
11	a_2 ranges over the 3^n marks of the $GF[3^n]$.
16	a_1 is a particular solution of $\phi(c) \equiv c + c^3 + \dots + c^{3^{n-1}} = 1$.
18	β ranges over the $3^n - 1$ multipliers in a rectangular table of the marks $\neq 0$ of the $GF[3^{3n}]$, those in the first row being the roots of $t^{1+3^n+3^{2n}} = 1$.
22	α ranges over the $3^n - 1$ marks $\neq 0$ of the $GF[3^n]$.

Thus for system (12), with the canonical forms 9–11,

$$\frac{2(3^{2n}-1)}{2(3^{2n}-1)}+\frac{2(3^{2n}-1)}{2(3^n+1)}+\frac{2(3^{2n}-1)}{2}\cdot 3^n=3^{3n}.$$

18. Aside from 17 and 18, all the canonical forms and automorphs in the Table have their coefficients in the initial field $GF[3^n]$. A similar treatment of 17 and 18 for $n=1$ will be given for illustration. We may set $s=1$; then the roots of (23) belong to the exponent 13. As the three canonical forms, we may take (14) with $s=1$ and

$$(40)\qquad (a_1,\ a_2,\ a_3)=(-1,\ -1,\ -1),\ (0,\ 0,\ 0),\ (0,\ 0,\ -1).$$

Indeed, for these, β given by (27) equals 0, $-i^2+1$, $-i^2-1$, respectively; while $(-i^2-1)/(-i^2+1)=1-i$ belongs to the exponent 26, so that no two of the forms given by (40) are equivalent. For $t=i$, (28) gives

$$(41)\qquad U\colon\ x_1'=x_2+x_3,\ x_2'=x_3,\ x_3'=x_1 \qquad (U^{13}=I).$$

For $\beta=1-i^2$, we get $\beta^{18}=-1-i^2$. Hence (29), for $t=-1-i^2$, gives

$$(42)\qquad V\colon\ x_1'=x_1+x_3,\ x_2'=x_1-x_2,\ x_3'=-x_1-x_2 \qquad (V^3=I),$$

an automorph of $x_1^2x_2-x_1^3x_2+x_2^2x_3+x_3^2x_1$. Hence (41) and (42) generate the group (32) of the system (14); its 39 transformations are the automorphs of (40_1). The 3 automorphs of (40_2) are the powers of V. The 3 automorphs of (40_3) are obtained similarly.

In (26′) for $n=1$, set $\xi=\beta^4y$. For β a square, $\beta^{13}=1$, we find that the form vanishes for 13 values of y, since $y^{12}+y^8+y^2+1$ divides $y^{26}-1$, modulo 3. For $\beta^{13}=-1$, the form vanishes for 7 values of y, since $-y^{12}+y^8+y^2+1$ and $y^{26}-1$ have the greatest common divisor y^6-y^2-1. Hence the forms (40) vanish for exactly 1, 7 and 13 respective sets of values x_i, modulo 3, since $1-i^2$ is a not-square and $-1-i^2$ a square in the $GF[3^3]$.

As the basis for a similar treatment for $n>1$, we note that, at least when $n=1, 2, 3, 4$, a root i of (23) is a primitive root in the $GF[3^{3n}]$ when s is a primitive root in the $GF[3^n]$.

19. Instead of forms 9, 10, 11, we may, on the basis of § 7, employ

Canonical form	Automorphs	Number
$yy^{3^n}x_2$	T_t, T_tL, defined by (19′), (20′)	$2(3^{2n}-1)$
$yy^{3^n}x_2+x_2^3$	Preceding with $t^{3^n+1}=1$	$2(3^n+1)$
$yy^{3^n}x_2+y^3+y^{3^{n+1}}+a_2x_2^3$	I, L	2

Hence under transformations with coefficients in a higher field, types 9–11, 17, 18 reduce to types analogous to 19–22.

20. Theorem. *Every ternary cubic form in the* $GF[3^n]$ *can be reduced, by a ternary linear transformation rational or irrational with respect to the given field, to one and but one of the eleven ultimate canonical forms* 1, 2, 3, 12, 14, 15, 19–23.

For systems (12), (14) and (15), the invariant Q is not zero. In view of § 19, the ultimate canonical forms are 19–22. For the remaining systems (11), (13) and (16), $Q = 0$, so that reduction to 19–22 is impossible; while (16) alone is reducible to 23. As in §§ 6, 8, no form (13) is reducible to a form (11) by a transformation of modulus 3. In the reduction of forms (11), in § 13, the case $a_3 = 0$ led to types 1, 2. For $a_3 \neq 0$, conditions (35′) may be satisfied for α and z in a higher field, when we set $a_3' = 1$, $a_2' = 0$. After the determination of α^3 as a fourth root of a_3^{-1}, α^3 follows rationally; the determination of z requires the solution of a cubic. Hence (11) with $a_3 \neq 0$ can be reduced to type 3 by a transformation in the $GF[3^{12n}]$.

In the reduction of forms (13) in § 15, the case $a_3 \neq 0$ led to type 14, the case $a_3 = a_2 = 0$ to type 12. For $a_3 = 0$, $a_2 \neq 0$, we may make $a_2 = -1$, $a_1 = 0$ by using a quadratic and a cubic irrationality; thus in the $GF[3^{6n}]$ we reach type 15.

The present list of ultimate forms for modulus 3 differs from Gordan's list of non-modular forms. His C_2, C_8, C_{10} are here equivalent to 23; C_1 to 22, C_3 to 21, C_4 to 20, C_5 to 19, C_6 to 14, C_7 to 1, C_9 to 12, while none correspond to 2, 3, 15.

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